Name——————————————, ID ————————

Exam I, MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Let D be a subgroup of $(Z_6, +)$ with 2 elements. Find all left cosets of D.

Trivial calculations, all of you got it right

(ii) Let $(A, *)$ be an abelian group with 30 elements. Suppose that A has a subgroup of order 3 and it has a cyclic subgroup of order 10. Prove that A has a unique subgroup of order 15.

Sketch: Let G be a subgroup of order 3. Since 3 is prime, G is cyclic. Hence $G =< a >$ for some $a \in A$ and thus $|a| = 3$. Let $F =< c >$ be a cyclic group of order 10. Hence $|c| = 10$. Since $gcd(3, 10) = 1$ and $a * c = c * a$, we know (Class NOTES), $|a * c| = 30$. Let $d = a * c$. Hence $A =< c >$. Since $15 \mid 30$ and A is cyclic, we know that A has a unique subgroup of order 15.

(iii) Given A is a cyclic group with 24 elements. Let $D = \{b \in A \mid A =< b >\}$. Find $|D|$. Assume that $A =< a >$ for some $a \in A$. Find all positive integers k such that $|a^k| = 8$.

Sketch: We know $|D| = \phi(24) = 12$. We know $gcd(24, k)$ must be 3. Hence, $k = 3, 9, 15, 21$ (exactly $\phi(8) = 4$ different elements)

(iv) Let $(A, *)$ be a group and $F = \{b \in A \mid b * a = a * b$ for every $a \in A\}$. Prove that F is a nonempty set, then prove that F is a subgroup of A.

Sketch: Since $e * w = w * e = w$ for every $w \in A$, $e \in F$ and thus F nonempty. Let $x, y \in F$. We show $x^{-1} * y \in F$ (i.e., we show that $x^{-1} * y$ commute with every element in A). Let $s \in A$. We show $x^{-1} * y * s = s * x^{-1} * y$. Since $x \in F$(i.e., $x * g = g * x$ for every $g \in A$). we know that $x^{-1} * s = s * x^{-1}$ (by HW). Hence $x^{-1} * y * s = x^{-1} * s * y = s * x^{-1} * y$. Thus $x^{-1} * y \in F$

(v) Let $(A, *)$ be a cyclic group with $n < \infty$ elements. Choose two positive integers say m, k such that $m \mid n$ and $k \mid m$ (hence $k \mid n$). Let F be a subgroup of A with m elements and let L be a subgroup of A with k elements.

a. Prove that $L \subset F$.

Sketch: Since A is cyclic and $k \mid n$, A has UNIQUE (stare at UNIQUE) subgroup with k elements, say W. Hence $L = W$. Since A is cyclic, F is the unique cyclic subgroup of A with m elements. Since F is cyclic and $k \mid m$, F has unique cyclic subgroup with k elements, say H, Hence H is also a subgroup of A with k elements (note $H < F < A$). Hence $K = W = L$. Thus $L \subset F$.

b. Assume $n = 12$, $m = 6$. Choose $d \in A$ such that $d \notin F$. Prove that $|d| = 4$ or 12.

Sketch: Let $h = |d|$. Then $h \mid 12$. Since A is cyclic, $G = \{e, d, ..., d^{h-1}\}$ is the unique subgroup of A with h elements. If $h = 1, 2, 3, 6$, then $h \mid m$ and $G \subset F$ by (a). Since $d \notin F$, $G \not\subseteq F$. Thus $h = 4 or 12$.

(vi) Let $(A, *)$ be a finite abelian group with 36 elements and let W be a subgroup of A with 9 elements. Suppose $a \in A$ such that $|a| = 2$. Let $M = a * W$ (so M is a left coset of W). Prove that $W \cup M$ is a subgroup of A with exactly 18 elements.

Sketch: Let $H = W \cup M$. We know that $W \cap M = \{\}$ and we Know $|W| = |M| = 9$. Hence $|H| = 18$. Since H is finite set, we only need to show that H is closed. Let $x, y \in H$. We consider three cases:

case 1: $x, y \in W$. Then clearly $x * y \in W$ (Since W is a group). Case two: $x, y \in M$. Then $x = a * w_1, y = a * w_2$ for some $w_1, w_2 \in W$. Thus $x * y = a * w_1 * a * w_2 = a^2 * w_1 * w_2$ (since A is abelian) $= e * w_1 * w_2 = w_1 * w_2 \in W \subset H$. Case three $x \in W, y \in M$. Hence, again, $y = a * w$ for some $w \in W$. Thus $x * y = x * a * w = a * x * w \in M$ (since $x * w \in W$). We are done

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

Name————————————, ID ——————

Exam II, MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. (i) Let H be a subgroup of A such that H has exactly two left cosets. Prove that H is normal in A.

Sketch: Let $a \in A$. If a $\in H$, then $a * H = H * a$. If $a \notin H$, then we know that $A = H \cup (a * H) = H \cup (H * a)$ and $H \cap (a * H) = H \cap (H * a) = \emptyset$, and thus $a * H = H * a$

(ii) Prove that S_3 has a normal subgroup of order 3.

Sketch: Since $|A_3| = 3$ and $\frac{|S_3|}{|A_3|} = 2$, we conclude that A_3 has exactly 2 distinct left cosets, and thus A_3 is normal in S_3 by (i) (or just by class note).

(iii) Given $F : (Z_8, +) \to (Z_6, +)$ is a non-trivial group homomorphism. Find $Ker(F)$ and $Range(F)$.

Sketch: All of you got it right!!!

(iv) We know that if H is a normal subgroup of a group A such that $|A/H|$ is finite, then A need not be finite. Let H be a finite normal subgroup of A such that $|A/H|$ is finite, prove that A is finite.

sketch Let $n = |A/H| = |A|/|H|$. Let $m = |H|$. Hence $|A| = mn$

(v) Given $F : (A, *) \to (B, \square)$ is a group homomorphism such that $F(v) = u$ for some $v \in A$. Prove that $F^{-1}(u) = v * Ker(F)$ (Note that$F^{-1}(u)$ is the set $\{a \in A | F(a) = u\}$.

Sketch: Let $K : A/Ker(F) \to Image(F)$, given by $K(a * Ker(F)) = F(a)$. We know that K is a group isomorphism. Thus $F^{-1}(u) = v * Ker(F)$.

(vi) Find the order of the element $(1\ 4\ 5)\ O\ (2\ 5\ 6\ 1) \in S_6$.

Sketch : Trivial

(vii) Is $(1\ 4\ 5)\ o\ (4\ 7\ 3\ 1)$ Even or Odd?

Sketch Trivial

(viii) Is the group $U(24)$ isomorphic to $(Z_8, +)$? explain.

sketch: (Note that each group is with 8 elements). Since 24 is not of the form $2p^m$ for some odd prime p, U(24) is not cyclic but Z_8 is cyclic. Hence they are not isomorphic

(ix) Let $H = Z_4 \times Z_4$, $K = Z_2 \times Z_8$. Then H and K are both abelian groups with 16 elements. However, show that H is not isomorphic to K.

sketch: K has an element of order 8 (for example (0, 1)) but each element in H is of order ≤ 4. So they cannot be isomorphic

(x) Let $F : A \to A$ be a group homomorphism such that $F(a) = a^{-1}$ for every $a \in A$. Prove that A is an abelian group

Sketch: Let $a, b \in A$. Then $F(a^{-1} * b^{-1}) = (a^{-1} * b^{-1})^{-1} = b * a$. Since F is a group homomorphism, $F(a^{-1} * b^{-1}) = F(a^{-1}) * F(b^{-1}) = a * b$. Thus $a * b = b * a$ (if you show $a^{-1} * b^{-1} = b^{-1} * a^{-1}$ is OK too by HW problem)

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

Name——————————————, ID ————————

Final Exam , MTH 320, Fall 2015

Ayman Badawi

QUESTION 1. 1) Prove that A_5 has a cyclic subgroups of order 6 and a cyclic subgroup of order 5 but it has no subgroups of order 30.

2) Let A be an abelian group of order 27 such that each element of A different from e has order 3. Up to isomorphism classify all such groups (i.e., up to isomorphism, find all possibilities of such A)

3) Let A be an abelian group with 75 element. If A has a cyclic subgroup of order 25, then prove that A is cyclic.

4) Let A be an abelian group with 100 elements such that A has no cyclic subgroups of order 25 and it has no cyclic subgroups of order 4. Prove that A has exactly 24 elements each is of order 5. How many elements of order 10 does A have?

5) Let A be a group with 60 elements. Assume that A has a normal subgroup B with 5 elements. Prove that B is the only subgroup of A with 5 elements.

6) It is clear that $(Z, +)$ is a normal subgroup of $(Q, +)$. Let $H = Q/Z$. Then we know that H is a group. Let $a = \frac{5}{7} + Z, b = \frac{1}{6} + Z \in H$. Find $|a|$ and $|b|$. Show that H has a cyclic subgroup with 21 elements.

7) Let $F : Z_{28} \to Z_7$ be a nontrivial group homomorphism. Find Range(F) and Ker(T).

8) Let A be a group with 77 elements. Prove that A is not simple.

9) Show that $U(45)$ is not group-isomorphic to $Z_2 \times Z_2 \times Z_6$.

10) Let $F : (P_2, +) \to (R, +)$ such that $F(f(x)) = \int_0^1 f(x)\, dx$. Then it is easy to see that F is a group homomorphism (do not show that). Note that P_2 is the set of all polynomials of degree strictly less than 2, i.e., P_2 consists of all constants and all polynomials of degree 1. Then

a) Find Ker(F)

b) Let $D = \{f(x) \in P_2 | F(f(x)) = \sqrt{3}\}$. Find the set D.

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com